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# COMPATIBILITY CONDITIONS OF SMALL DEFORMATIONS AND STRESS FUNCTIONS

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The Saint Venant compatibility conditions and the Maxwell and Morera stress functions have been known since the last century and are presented in textbooks on the theory of elasticity, but with insufficient clarity. There is extensive literature on this subject (see, e.g., [1–23]) in which the number of independent compatibility conditions, the generality and completeness of stress functions, and formulations of elastic problems in terms of stresses are discussed.

In the present paper, it is shown that there are 17 equivalent forms, with three compatibility conditions of small strains in each, to which correspond 17 forms of representation of stresses via three stress functions. Any of the 17 forms of stress representation is proved to be the general and complete solution of equilibrium equations. A new formulation of elastic equations in terms of stresses is given, and it is shown that 289 versions of systems of equations for three stress functions are possible. It follows from the results of the study that it is admissible to formulate the boundary-value problem for a system of six equations for three stress functions and three displacements instead of the formulation of the problem for displacement and stress equations.

In the absence of volumetric forces, we have the following equilibrium equations in the Cartesian rectangular coordinates $x_1$, $x_2$, and $x_3$:

$$\partial_j \sigma_{ij} = 0, \tag{1}$$

to which correspond the stresses $\sigma_{ij} = \sigma_{ji}$ expressed via the tensor of stress functions $\gamma_{pq} = \gamma_{qp}$ [3–4]:

$$\sigma_{ij} = \varepsilon_{imp}\varepsilon_{jnq}\partial_{mn}\gamma_{pq}. \tag{2}$$

Here $\partial_j$ are the derivatives with respect to the $x_j$ coordinate and $\varepsilon_{imp}$ are the Levi-Civita symbols; the repeat indices imply summation. The small strains $\varepsilon_{ij} = \varepsilon_{ji}$ are expressed via the displacements $u_j$ as follows:

$$\varepsilon_{ij} = (\partial_j u_i + \partial_i u_j)/2. \tag{3}$$

The compatibility conditions are usually written in the form

$$s_{ij} = \varepsilon_{imp}\varepsilon_{jnq}\partial_{mn}\varepsilon_{pq} = 0. \tag{4}$$

Note that for the so-called incompatibility tensor $s_{ij} = s_{ji}$, the equalities

$$\partial_j s_{ij} = \varepsilon_{imp}\varepsilon_{jnq}\partial_{mnj}\varepsilon_{pq} \equiv 0$$

are satisfied.

Let us introduce the compatibility conditions in a way different from that used in the textbooks on elasticity theory. We write Eq. (3) in more detail as follows:

$$\begin{aligned}
&\partial_1 u_1 = \varepsilon_{11} = \varepsilon_1, \qquad \partial_3 u_2 + \partial_2 u_3 = 2\varepsilon_{23} = \sqrt{2}\varepsilon_4,\\
&\partial_2 u_2 = \varepsilon_{22} = \varepsilon_2, \qquad \partial_3 u_1 + \partial_1 u_3 = 2\varepsilon_{13} = \sqrt{2}\varepsilon_5,\\
&\partial_3 u_3 = \varepsilon_{33} = \varepsilon_3, \qquad \partial_2 u_1 + \partial_1 u_2 = 2\varepsilon_{12} = \sqrt{2}\varepsilon_6.
\end{aligned} \tag{5}$$

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If the strains $\varepsilon_i$ are given, then for three displacements $u_i$, there are six equations (5). Since the number of equations is greater than the number of unknowns, we have to check the compatibility of system (5).

We choose linearly independent subsystems of three equations from (5) from which three displacements are defined and write the remaining three equations. The following versions of grouping of Eqs. (5) are possible, with three equations in each:

$$
\begin{array}{llll}
\text{No. 1. } (1,2,3;\ 4,5,6), & \text{No. 6. } (\underline{1,3,5};\ 2,4,6), & \text{No. 11. } (\underline{2,3,4};\ 1,5,6), & \text{No. 16. } (2,5,6;\ 1,3,4), \\
\text{No. 2. } (1,2,4;\ 3,5,6), & \text{No. 7. } (1,3,6;\ 2,4,5), & \text{No. 12. } (2,3,5;\ 1,4,6), & \text{No. 17. } (3,4,5;\ 1,2,6), \\
\text{No. 3. } (1,2,5;\ 3,4,6), & \text{No. 8. } (1,4,5;\ 2,3,6), & \text{No. 13. } (2,3,6;\ 1,4,5), & \text{No. 18. } (3,4,6;\ 1,2,5), \\
\text{No. 4. } (\underline{1,2,6};\ 3,4,5), & \text{No. 9. } (1,4,6;\ 2,3,5), & \text{No. 14. } (2,4,5;\ 1,3,6), & \text{No. 19. } (3,5,6;\ 1,2,4), \\
\text{No. 5. } (1,3,4;\ 2,5,6), & \text{No. 10. } (1,5,6;\ 2,3,4), & \text{No. 15. } (2,4,6;\ 1,3,5), & \text{No. 20. } (4,5,6;\ 1,2,3).
\end{array}
\tag{6}
$$

Here the first figures in the parentheses denote the row (strain) numbers that are considered independent. For the underlined versions, the rows (strains) are linearly dependent, and the determinants of these subsystems are equal to zero. Thus, 17 versions of (6) of subsystems remain, from which one can find three displacements. In this case, instead of (5), we have 17 versions of systems of the form

$$
A_1 u = a, \quad B_1 u = b, \quad |A_1| \neq 0. \tag{7}
$$

For compatibility of system (7), the corresponding bordering minors of the augmented matrix must be zero [24]. For example, for version No. 1 [see (6)], the matrices $A_1$ and $B_1$ are of the form

$$
A_1 = \begin{bmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{bmatrix}. \tag{8}
$$

Let us equate the bordering minors of the augmented matrix to zero:

$$
M_{(1,2,3,4)} = \begin{vmatrix} \partial_1 & 0 & 0 & \varepsilon_1 \\ 0 & \partial_2 & 0 & \varepsilon_2 \\ 0 & 0 & \partial_3 & \varepsilon_3 \\ 0 & \partial_3 & \partial_2 & \sqrt{2}\varepsilon_4 \end{vmatrix} = \partial_1(\partial_{23}\sqrt{2}\varepsilon_4 - \partial_{33}\varepsilon_2 - \partial_{22}\varepsilon_3) = 0,
$$

$$
M_{(1,2,3,5)} = \begin{vmatrix} \partial_1 & 0 & 0 & \varepsilon_1 \\ 0 & \partial_2 & 0 & \varepsilon_2 \\ 0 & 0 & \partial_3 & \varepsilon_3 \\ \partial_3 & 0 & \partial_1 & \sqrt{2}\varepsilon_5 \end{vmatrix} = \partial_2(\partial_{13}\sqrt{2}\varepsilon_5 - \partial_{33}\varepsilon_1 - \partial_{11}\varepsilon_3) = 0, \tag{9}
$$

$$
M_{(1,2,3,6)} = \begin{vmatrix} \partial_1 & 0 & 0 & \varepsilon_1 \\ 0 & \partial_2 & 0 & \varepsilon_2 \\ 0 & 0 & \partial_3 & \varepsilon_3 \\ \partial_2 & \partial_1 & 0 & \sqrt{2}\varepsilon_6 \end{vmatrix} = \partial_3(\partial_{12}\sqrt{2}\varepsilon_6 - \partial_{22}\varepsilon_1 - \partial_{11}\varepsilon_2) = 0.
$$

The factors $\partial_i$ in (9) that are not equal to zero can be omitted, and we can write the compatibility conditions in the form

$$
\partial_{23}\sqrt{2}\varepsilon_4 = \partial_{33}\varepsilon_2 + \partial_{22}\varepsilon_3, \quad \partial_{13}\sqrt{2}\varepsilon_5 = \partial_{33}\varepsilon_1 + \partial_{11}\varepsilon_3, \quad \partial_{12}\sqrt{2}\varepsilon_6 = \partial_{22}\varepsilon_1 + \partial_{11}\varepsilon_2, \quad C_1 b = D_1 a,
$$

$$
C_1 = \begin{bmatrix} \partial_{23} & 0 & 0 \\ 0 & \partial_{13} & 0 \\ 0 & 0 & \partial_{12} \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & \partial_{33} & \partial_{22} \\ \partial_{33} & 0 & \partial_{11} \\ \partial_{22} & \partial_{11} & 0 \end{bmatrix}. \tag{10}
$$

It is seen from (7), (8), and (10) that the relation $C_1 B_1 = D_1 A_1$ is fulfilled.

Thus, the compatibility conditions of system (7) are of the form $C_1 b = D_1 a$, where $C_1$ and $D_1$ are the operators with a minimum possible order of derivatives. Note that $C_1 B_1 = D_1 A_1$. For equations with constant coefficients of the matrices, $C_1$ and $D_1$ are obtained from the requirement of zero bordering minors of the augmented matrix [see (9) and (10)]. If $C_1$ and $D_1$ have a common factor, it should be omitted [see (9) and (10)], because, apparently, one should use $C_1$ and $D_1$ of minimum degree with respect to $\partial_i$, and it does not make sense to increase the order of these operators. On the other hand, if $C_1 b = D_1 a$ and $C_1 B_1 = D_1 A_1$, there exists $u$ such that $a = A_1 u$ and $b = B_1 u$.

Indeed, the general solution of the linear system (7) is of the form $u = u^{(1)} + u^{(0)}$, where $u^{(1)}$ is the particular solution (if it exists) of the nonhomogeneous system $A_1 u^{(1)} = a$ and $B_1 u^{(1)} = b$, and $u^{(0)}$ is the general solution of the homogeneous equations $A_1 u^{(0)} = 0$ and $B_1 u^{(0)} = 0$. In essence, the second equation $B_1 u^{(1)} = b$ is the compatibility condition. We exclude $u^{(1)}$ from it. Let $C_1 B_1 = D_1 A_1$ ($C_1$ and $D_1$ are the operators of minimum possible order). After that, having multiplied $B_1 u^{(1)} = b$ by $C_1$, we obtain successively $C_1 B_1 u^{(1)} = C_1 b$, $D_1 A_1 u^{(1)} = C_1 b$, and $D_1 a = C_1 b$, i.e., the last relation is the necessary compatibility condition. In addition, it is the sufficient compatibility condition of the system $A_1 u^{(1)} = a$ and $B_1 u^{(1)} = b$. In [25], the general solution of the equation $D_1 a = C_1 b$ was proved to be of the following form: $a = A_1 \varphi$, $b = B_1 \varphi + \psi$, and $C_1 \psi = 0$ with $C_1 B_1 = D_1 A_1$, i.e., for any concrete $a$ and $b$, subject to the condition $D_1 a = C_1 b$, there are $\varphi$ and $\psi$ such that $a$ and $b$ are represented in the above form. Substituting these $a$ and $b$ into our system, we have $A_1 u^{(1)} = A_1 \varphi$ and $B_1 u^{(1)} = B_1 \varphi + \psi$. From the first equation, we then have $u^{(1)} = \varphi$, and from the second one, we obtain $\psi = 0$. Thus, there is no need to write the unnecessary functions $\psi$ and $C_1 \psi = 0$, and the relation $D_1 a = C_1 b$ is the necessary and sufficient compatibility condition of the system $A_1 u^{(1)} = a$ and $B_1 u^{(1)} = b$.

We shall perform all manipulations for the remaining versions of (6) and write appropriate compatibility conditions that permit us to find the matrices $C_1$ and $D_1$. As a result, we obtain

No. 1. $(1,2,3;\ 4,5,6)$

$$\partial_{23}\sqrt{2}\varepsilon_4 = \partial_{33}\varepsilon_2 + \partial_{22}\varepsilon_3,$$
$$\partial_{13}\sqrt{2}\varepsilon_5 = \partial_{33}\varepsilon_1 + \partial_{11}\varepsilon_3,$$
$$\partial_{12}\sqrt{2}\varepsilon_6 = \partial_{22}\varepsilon_1 + \partial_{11}\varepsilon_2;$$

No. 2. $(1,2,4;\ 3,5,6)$

$$\partial_{22}\varepsilon_3 = -\partial_{33}\varepsilon_2 + \partial_{23}\sqrt{2}\varepsilon_4,$$
$$\partial_{122}\sqrt{2}\varepsilon_5 = \partial_{223}\varepsilon_1 - \partial_{113}\varepsilon_2 + \partial_{112}\sqrt{2}\varepsilon_4,$$
$$\partial_{12}\sqrt{2}\varepsilon_6 = \partial_{22}\varepsilon_1 + \partial_{11}\varepsilon_2;$$

No. 3. $(1,2,5;\ 3,4,6)$

$$\partial_{11}\varepsilon_3 = -\partial_{33}\varepsilon_1 + \partial_{13}\sqrt{2}\varepsilon_5,$$
$$\partial_{112}\sqrt{2}\varepsilon_4 = -\partial_{223}\varepsilon_1 + \partial_{113}\varepsilon_2 + \partial_{122}\sqrt{2}\varepsilon_5,$$
$$\partial_{12}\sqrt{2}\varepsilon_6 = \partial_{22}\varepsilon_1 + \partial_{11}\varepsilon_2;$$

No. 4. $(\underline{1,2,6};\ 3,4,5)$

$$|A_1| = 0;$$

No. 5. $(1,3,4;\ 2,5,6)$

$$\partial_{33}\varepsilon_2 = -\partial_{22}\varepsilon_3 + \partial_{23}\sqrt{2}\varepsilon_4,$$
$$\partial_{13}\sqrt{2}\varepsilon_5 = \partial_{33}\varepsilon_1 + \partial_{11}\varepsilon_3,$$
$$\partial_{133}\sqrt{2}\varepsilon_6 = \partial_{233}\varepsilon_1 - \partial_{112}\varepsilon_3 + \partial_{113}\sqrt{2}\varepsilon_4;$$

No. 20. $(4,5,6;\ 1,2,3)$

$$2\partial_{23}\varepsilon_1 = -\partial_{11}\sqrt{2}\varepsilon_4 + \partial_{12}\sqrt{2}\varepsilon_5 + \partial_{13}\sqrt{2}\varepsilon_6,$$
$$2\partial_{13}\varepsilon_2 = \partial_{12}\sqrt{2}\varepsilon_4 - \partial_{22}\sqrt{2}\varepsilon_5 + \partial_{23}\sqrt{2}\varepsilon_6,$$
$$2\partial_{12}\varepsilon_3 = \partial_{13}\sqrt{2}\varepsilon_4 + \partial_{23}\sqrt{2}\varepsilon_5 - \partial_{33}\sqrt{2}\varepsilon_6;$$

No. 19. $(3,5,6;\ 1,2,4)$

$$\partial_{33}\varepsilon_1 = -\partial_{11}\varepsilon_3 + \partial_{13}\sqrt{2}\varepsilon_5,$$
$$\partial_{133}\varepsilon_2 = \partial_{122}\varepsilon_3 - \partial_{223}\sqrt{2}\varepsilon_5 + \partial_{233}\sqrt{2}\varepsilon_6,$$
$$\partial_{13}\sqrt{2}\varepsilon_4 = 2\partial_{12}\varepsilon_3 - \partial_{23}\sqrt{2}\varepsilon_5 + \partial_{33}\sqrt{2}\varepsilon_6;$$

No. 18. $(3,4,6;\ 1,2,5)$

$$\partial_{233}\varepsilon_1 = \partial_{112}\varepsilon_3 - \partial_{113}\sqrt{2}\varepsilon_4 + \partial_{133}\sqrt{2}\varepsilon_6,$$
$$\partial_{33}\varepsilon_2 = -\partial_{22}\varepsilon_3 + \partial_{23}\sqrt{2}\varepsilon_4,$$
$$\partial_{23}\sqrt{2}\varepsilon_5 = 2\partial_{12}\varepsilon_3 - \partial_{13}\sqrt{2}\varepsilon_4 + \partial_{33}\sqrt{2}\varepsilon_6;$$

No. 17. $(3,4,5;\ 1,2,6)$

$$\partial_{33}\varepsilon_1 = -\partial_{11}\varepsilon_3 + \partial_{13}\sqrt{2}\varepsilon_5,$$
$$\partial_{33}\varepsilon_2 = -\partial_{22}\varepsilon_3 + \partial_{23}\sqrt{2}\varepsilon_4,$$
$$\partial_{33}\sqrt{2}\varepsilon_6 = -2\partial_{12}\varepsilon_3 + \partial_{13}\sqrt{2}\varepsilon_4 + \partial_{23}\sqrt{2}\varepsilon_5;$$

No. 16. $(2,5,6;\ 1,3,4)$

$$\partial_{22}\varepsilon_1 = -\partial_{11}\varepsilon_2 + \partial_{12}\sqrt{2}\varepsilon_6,$$
$$\partial_{122}\varepsilon_3 = \partial_{133}\varepsilon_2 + \partial_{223}\sqrt{2}\varepsilon_5 - \partial_{233}\sqrt{2}\varepsilon_6,$$
$$\partial_{12}\sqrt{2}\varepsilon_4 = 2\partial_{13}\varepsilon_2 + \partial_{22}\sqrt{2}\varepsilon_5 - \partial_{23}\sqrt{2}\varepsilon_6;$$

No. 6. (1, 3, 5; 2, 4, 6)

$$|A_1| = 0;$$

No. 7. (1, 3, 6; 2, 4, 5)

$$\begin{aligned} \partial_{11}\varepsilon_2 &= -\partial_{22}\varepsilon_1 + \partial_{12}\sqrt{2}\varepsilon_6, \\ \partial_{113}\sqrt{2}\varepsilon_4 &= -\partial_{233}\varepsilon_1 + \partial_{112}\varepsilon_3 + \partial_{133}\sqrt{2}\varepsilon_6, \\ \partial_{13}\sqrt{2}\varepsilon_5 &= \partial_{33}\varepsilon_1 + \partial_{11}\varepsilon_3; \end{aligned}$$

No. 8. (1, 4, 5; 2, 3, 6)

$$\begin{aligned} \partial_{113}\varepsilon_2 &= \partial_{223}\varepsilon_1 + \partial_{112}\sqrt{2}\varepsilon_4 - \partial_{122}\sqrt{2}\varepsilon_5, \\ \partial_{11}\varepsilon_3 &= -\partial_{33}\varepsilon_1 + \partial_{13}\sqrt{2}\varepsilon_5, \\ \partial_{13}\sqrt{2}\varepsilon_6 &= 2\partial_{23}\varepsilon_1 + \partial_{11}\sqrt{2}\varepsilon_4 - \partial_{12}\sqrt{2}\varepsilon_5; \end{aligned}$$

No. 9. (1, 4, 6; 2, 3, 5)

$$\begin{aligned} \partial_{11}\varepsilon_2 &= -\partial_{22}\varepsilon_1 + \partial_{12}\sqrt{2}\varepsilon_6, \\ \partial_{112}\varepsilon_3 &= \partial_{233}\varepsilon_1 + \partial_{113}\sqrt{2}\varepsilon_4 - \partial_{133}\sqrt{2}\varepsilon_6, \\ \partial_{12}\sqrt{2}\varepsilon_5 &= 2\partial_{23}\varepsilon_1 + \partial_{11}\sqrt{2}\varepsilon_4 - \partial_{13}\sqrt{2}\varepsilon_6; \end{aligned}$$

No. 10. (1, 5, 6; 2, 3, 4)

$$\begin{aligned} \partial_{11}\varepsilon_2 &= -\partial_{22}\varepsilon_1 + \partial_{12}\sqrt{2}\varepsilon_6, \\ \partial_{11}\varepsilon_3 &= -\partial_{33}\varepsilon_1 + \partial_{13}\sqrt{2}\varepsilon_5, \\ \partial_{11}\sqrt{2}\varepsilon_4 &= -2\partial_{23}\varepsilon_1 + \partial_{12}\sqrt{2}\varepsilon_5 + \partial_{13}\sqrt{2}\varepsilon_6; \end{aligned}$$

No. 15 (2, 4, 6; 1, 3, 5)

$$\begin{aligned} \partial_{22}\varepsilon_1 &= -\partial_{11}\varepsilon_2 + \partial_{12}\sqrt{2}\varepsilon_6, \\ \partial_{22}\varepsilon_3 &= -\partial_{33}\varepsilon_2 + \partial_{23}\sqrt{2}\varepsilon_4, \\ \partial_{22}\sqrt{2}\varepsilon_5 &= -2\partial_{13}\varepsilon_2 + \partial_{12}\sqrt{2}\varepsilon_4 + \partial_{23}\sqrt{2}\varepsilon_6; \end{aligned} \qquad (11)$$

No. 14. (2, 4, 5; 1, 3, 6)

$$\begin{aligned} \partial_{223}\varepsilon_1 &= \partial_{113}\varepsilon_2 - \partial_{112}\sqrt{2}\varepsilon_4 + \partial_{122}\sqrt{2}\varepsilon_5, \\ \partial_{22}\varepsilon_3 &= -\partial_{33}\varepsilon_2 + \partial_{23}\sqrt{2}\varepsilon_4, \\ \partial_{23}\sqrt{2}\varepsilon_6 &= 2\partial_{13}\varepsilon_2 - \partial_{12}\sqrt{2}\varepsilon_4 + \partial_{22}\sqrt{2}\varepsilon_5; \end{aligned}$$

No. 13. (2, 3, 6; 1, 4, 5)

$$\begin{aligned} \partial_{22}\varepsilon_1 &= -\partial_{11}\varepsilon_2 + \partial_{12}\sqrt{2}\varepsilon_6, \\ \partial_{23}\sqrt{2}\varepsilon_4 &= \partial_{33}\varepsilon_2 + \partial_{22}\varepsilon_3, \\ \partial_{223}\sqrt{2}\varepsilon_5 &= -\partial_{133}\varepsilon_2 + \partial_{122}\varepsilon_3 + \partial_{233}\sqrt{2}\varepsilon_6; \end{aligned}$$

No. 12. (2, 3, 5; 1, 4, 6)

$$\begin{aligned} \partial_{33}\varepsilon_1 &= -\partial_{11}\varepsilon_3 + \partial_{13}\sqrt{2}\varepsilon_5, \\ \partial_{23}\sqrt{2}\varepsilon_4 &= \partial_{33}\varepsilon_2 + \partial_{22}\varepsilon_3, \\ \partial_{233}\sqrt{2}\varepsilon_6 &= \partial_{133}\varepsilon_2 - \partial_{122}\varepsilon_3 + \partial_{223}\sqrt{2}\varepsilon_5; \end{aligned}$$

No. 11. (2, 3, 4; 1, 5, 6)

$$|A_1| = 0.$$

Thus, we have obtained 17 versions of the compatibility conditions (11). Each version contains three compatibility conditions rather than six, as in the Saint Venant classical conditions (4). These 17 versions are equivalent, and, hence, correspond to the same initial (overdetermined) system (5) and are derived from one another if one solves them with respect to any triad of strains whose complementary determinant $|A_1|$ is not equal to zero.

We have to mention compatibility conditions that are different in form. These are the first six conditions in (11) and the conditions with the third derivatives, which were not previously known. As is seen from (11), there are only three conditions with the third derivatives, i.e., there are apparently only nine different conditions; we write them in the matrix form

$$\begin{array}{l} 1. \\ 2. \\ 3. \end{array} \quad \begin{bmatrix} \partial_{23} & 0 & 0 \\ 0 & \partial_{13} & 0 \\ 0 & 0 & \partial_{12} \end{bmatrix} \begin{bmatrix} \sqrt{2}\varepsilon_4 \\ \sqrt{2}\varepsilon_5 \\ \sqrt{2}\varepsilon_6 \end{bmatrix} = \begin{bmatrix} 0 & \partial_{33} & \partial_{22} \\ \partial_{33} & 0 & \partial_{11} \\ \partial_{22} & \partial_{11} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix},$$

$$\begin{array}{l} 4. \\ 5. \\ 6. \end{array} \quad \begin{bmatrix} 2\partial_{23} & 0 & 0 \\ 0 & 2\partial_{13} & 0 \\ 0 & 0 & 2\partial_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} -\partial_{11} & \partial_{12} & \partial_{13} \\ \partial_{12} & -\partial_{22} & \partial_{23} \\ \partial_{13} & \partial_{23} & -\partial_{33} \end{bmatrix} \begin{bmatrix} \sqrt{2}\varepsilon_4 \\ \sqrt{2}\varepsilon_5 \\ \sqrt{2}\varepsilon_6 \end{bmatrix}, \qquad (12)$$

$$\begin{array}{l} 7. \\ 8. \\ 9. \end{array} \quad \begin{bmatrix} 0 & -\partial_{133} & \partial_{122} \\ \partial_{233} & 0 & -\partial_{112} \\ -\partial_{223} & \partial_{113} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} 0 & \partial_{223} & -\partial_{233} \\ -\partial_{113} & 0 & \partial_{133} \\ \partial_{112} & -\partial_{122} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}\varepsilon_4 \\ \sqrt{2}\varepsilon_5 \\ \sqrt{2}\varepsilon_6 \end{bmatrix}.$$

The first six equations here are the Saint Venant conventional conditions (4), while the last three are additional conditions that were not known previously, but all together they are not compatibility conditions as equation Nos. 1–3 or 4–6 from (12). They just enter other triads of compatibility conditions. Below, we write these 17 triads of possible compatibility conditions [see (11) and (12)] as follows:

1. (1,2,3), 2. (4,5,6), 3. (1,9,3), 4. (2,7,6), 5. (2,9,3), 6. (8,1,6),
7. (2,1,6), 8. (1,2,8), 9. (3,7,5), 10. (3,1,5), 11. (3,8,2), 12. (9,1,5),
13. (9,2,4), 14. (3,1,7), 15. (3,8,4), 16. (2,1,7), 17. (3,2,4).

These triads of conditions can be obtained informally, by direct integration of Eqs. (5). For example, we have the first version (1,2,3). From the first three equations of (5), we then find by integration that

$$\begin{aligned} u_1^{(1)} &= \partial_1^{-1}\varepsilon_1 + \varphi_1(x_2, x_3), \\ u_2^{(1)} &= \partial_2^{-1}\varepsilon_2 + \varphi_2(x_1, x_3), \qquad \partial_j^{-1} = \int (\ldots)\, dx_j. \\ u_3^{(1)} &= \partial_3^{-1}\varepsilon_3 + \varphi_3(x_1, x_2), \end{aligned} \tag{13}$$

After substitution of (13) into the last three equations of (5), we obtain

$$\begin{aligned} \sqrt{2}\varepsilon_4 &= \partial_3(\partial_2^{-1}\varepsilon_2 + \varphi_2(x_1, x_3)) + \partial_2(\partial_3^{-1}\varepsilon_3 + \varphi_3(x_1, x_2)), \\ \sqrt{2}\varepsilon_5 &= \partial_3(\partial_1^{-1}\varepsilon_1 + \varphi_1(x_2, x_3)) + \partial_1(\partial_3^{-1}\varepsilon_3 + \varphi_3(x_1, x_2)), \\ \sqrt{2}\varepsilon_6 &= \partial_2(\partial_1^{-1}\varepsilon_1 + \varphi_1(x_2, x_3)) + \partial_1(\partial_2^{-1}\varepsilon_2 + \varphi_2(x_1, x_3)). \end{aligned} \tag{14}$$

Relations (14) are precisely the compatibility conditions, i.e., if $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ are independent functions, then $\varepsilon_4$, $\varepsilon_5$, and $\varepsilon_6$ cannot be arbitrary functions and should have the form (14) for system (5) be compatible. To derive conditions (10) from (14), we differentiate each equation in (14) with respect to $\partial_{23}$, $\partial_{13}$, and $\partial_{12}$, respectively. Clearly, the functions $\varphi_1$, $\varphi_2$, and $\varphi_3$ vanish after differentiation, and we obtain three equations (10), as it should be, i.e., the compatibility conditions (14) or (10) are only three in number, rather than six, as is considered traditionally [see (4)]. We can act similarly with other versions. A similar approach and the derivation of equations of the form (14) are given in [9, 14, 15].

The general solution of the linear equations (5) consists of the partial solution of nonhomogeneous equations and the general solution of homogeneous ones, i.e., when $\varepsilon_i = 0$. Let us find the last solution. From (13) and (14), for $\varepsilon_i = 0$, we obtain

$$\begin{aligned} u_1^{(0)} &= \varphi_1^{(0)}(x_2, x_3), \qquad \partial_3\varphi_2^{(0)}(x_1, x_3) + \partial_2\varphi_3^{(0)}(x_1, x_2) = 0, \\ u_2^{(0)} &= \varphi_2^{(0)}(x_1, x_3), \qquad \partial_3\varphi_1^{(0)}(x_2, x_3) + \partial_1\varphi_3^{(0)}(x_1, x_2) = 0, \\ u_3^{(0)} &= \varphi_3^{(0)}(x_1, x_2), \qquad \partial_2\varphi_1^{(0)}(x_2, x_3) + \partial_1\varphi_2^{(0)}(x_1, x_3) = 0. \end{aligned} \tag{15}$$

It follows from (15) that $\varphi_i^{(0)}$ can be only the following differential functions:

$$\begin{aligned} \varphi_1^{(0)}(x_2, x_3) &= \alpha_1 + \alpha_{12}x_2 + \alpha_{13}x_3, \\ \varphi_2^{(0)}(x_1, x_3) &= \alpha_2 + \alpha_{21}x_1 + \alpha_{23}x_3, \qquad \alpha_{ij} + \alpha_{ji} = 0. \\ \varphi_3^{(0)}(x_1, x_2) &= \alpha_3 + \alpha_{31}x_1 + \alpha_{32}x_2, \end{aligned} \tag{16}$$

Expressions (16) determine the displacements of a body as a solid. Formulas (13) and (16) are the representations of displacements via strains (and there is no need to use the Cesàro formulas), and the functions $\varphi_i$ are found from (14). Thus, the whole arbitrariness in the integration of system (5) is determined by the linear functions (16). Therefore, it suffices to find only a partial solution of the nonhomogeneous system (5).

Now let us consider Eqs. (1). We denote $\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sqrt{2}\sigma_{23}, \sqrt{2}\sigma_{13}, \sqrt{2}\sigma_{12})$ and write (1) and (3) and the Hooke generalized law in matrix form

$$C\sigma = 0, \quad \sigma = A\varepsilon, \quad \varepsilon = C'u, \tag{17}$$

where $A = A'$ is the elasticity matrix,

$$C = \begin{bmatrix} \partial_1 & 0 & 0 & 0 & (1/\sqrt{2})\partial_3 & (1/\sqrt{2})\partial_2 \\ 0 & \partial_2 & 0 & (1/\sqrt{2})\partial_3 & 0 & (1/\sqrt{2})\partial_1 \\ 0 & 0 & \partial_3 & (1/\sqrt{2})\partial_2 & (1/\sqrt{2})\partial_1 & 0 \end{bmatrix}; \tag{18}$$

the prime denotes transposition.

Since Eqs. (1) are three in number, and the unknowns $\sigma_i$ are six in number, grouping the quantities $\sigma_i$ by three, we can rewrite (1) as the compatibility conditions [see (10)]

$$A_1'v = B_1'w, \quad |A_1'| \neq 0. \tag{19}$$

Here $A_1'$ and $B_1'$ are the transposes $A_1$ and $B_1$ from (7). The possible versions of grouping of the quantities $\sigma_i$ by three are given above [see (6)]. It follows from the relation $D_1A_1 = C_1B_1$ that

$$A_1'D_1' = B_1'C_1'. \tag{20}$$

With allowance for (20), we find the general solution [25] of Eqs. (19) via three arbitrary stress functions $\varphi_1$, $\varphi_2$, and $\varphi_3$:

$$v = D_1'\varphi, \qquad w = C_1'\varphi. \tag{21}$$

In [25], the general solution of Eqs. (19) is as follows: $v = D_1'\varphi$, $w = C_1'\varphi + \psi$, and $B_1'\psi = 0$, but if all operators are linear and the relation $C_1' \operatorname{Ker} D_1' = \operatorname{Ker} B_1'$ (Ker is the kernel of the operator) is satisfied, the general solution is as follows: $v = D_1'\varphi$, $w = C_1'\varphi$.

A direct verification shows that the condition $C_1' \operatorname{Ker} D_1' = \operatorname{Ker} B_1'$ for the operators from (19) and (20) is satisfied, thereby ensuring the generality of the solution of (21). The generality of the solution of the Maxwell and Morera solutions [see solution Nos. 1 and 20 in (22) below] is proved in [1].

Let us check the satisfaction of the condition $C_1' \operatorname{Ker} D_1' = \operatorname{Ker} B_1'$, for example, for solution No. 2 from (22). The kernel of the operator $B_1'$ is found from the equations $\partial_3\psi_2 + \partial_2\psi_3 = 0$, $\partial_1\psi_3 = 0$, and $\partial_3\psi_1 + \partial_1\psi_2 = 0$ whose solution is of the form $\psi_1 = g_3(x_1, x_2) - \partial_1 f_3(x_1, x_2)x_3$, $\psi_2 = f_3(x_1, x_2) - \partial_2 f_1(x_2, x_3)$ and $\psi_3 = \partial_3 f_1(x_2, x_3)$, where $f_i$ and $g_3$ are arbitrary functions of the corresponding arguments. The kernel of the operator $D_1'$ is determined from the equations $\partial_{223}\varphi_2 + \partial_{22}\varphi_3 = 0$, $-\partial_{33}\varphi_1 - \partial_{113}\varphi_2 + \partial_{11}\varphi_3 = 0$, and $\partial_{23}\varphi_1 + \partial_{112}\varphi_2 = 0$, whose solution is as follows:

$$\varphi_1 = 2\partial_{11}\gamma_2(x_1, x_3) - \partial_{11}[\alpha_2(x_1, x_3) + \beta_2(x_1, x_3)x_2] - \partial_{11}\gamma_3(x_1, x_2)x_3 + h_3(x_1, x_2),$$

$$\varphi_2 = -\partial_3\gamma_2(x_1, x_3) + \partial_3[\alpha_2(x_1, x_3) + \beta_2(x_1, x_3)x_2] + \gamma_3(x_1, x_2) - \alpha_1(x_2, x_3) - \beta_1(x_2, x_3)x_1,$$

$$\varphi_3 = \partial_{33}\gamma_2(x_1, x_3) + \partial_3[\alpha_1(x_2, x_3) + \beta_1(x_2, x_3)x_1].$$

Here $\alpha_i$, $\beta_i$, and $\gamma_i$, $h_3$ are arbitrary functions of the corresponding arguments. Multiplying the functions $\varphi_i$ by the operator $C_1'$, we have

$$\begin{aligned}
p_1 &= \partial_{22}\varphi_1 = \partial_{22}[h_3(x_1, x_2) - \partial_{11}\gamma_3(x_1, x_2)x_3], \\
p_2 &= \partial_{122}\varphi_2 = \partial_{122}\gamma_3(x_1, x_2) - \partial_{22}\beta_1(x_2, x_3), \\
p_3 &= \partial_{12}\varphi_3 = \partial_{23}\beta_1(x_2, x_3).
\end{aligned}$$

Having denoted $\partial_{22}h_3(x_1, x_2) = g_3(x_1, x_2)$, $\partial_{122}\gamma_3(x_1, x_2) = f_3(x_1, x_2)$, and $\partial_2\beta_1(x_2, x_3) = f_1(x_2, x_3)$, we see that the relations for $p_i$ and $\psi_i$ coincide. This points to the fact that the condition $C_1' \operatorname{Ker} D_1' = \operatorname{Ker} B_1'$ is satisfied, and solution No. 2 from (22) is general. Similarly, one can check the satisfaction of this condition for the remaining solutions of (22).

In accordance with the versions from (6), we now write 17 forms of the general solution of the equilibrium equations (1), i.e., 17 forms of stress representations via three stress functions, as follows:

No. 1. (1,2,3; 4,5,6) (Maxwell solution)

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} 0 & \partial_{33} & \partial_{22}\\ \partial_{33} & 0 & \partial_{11}\\ \partial_{22} & \partial_{11} & 0\\ -\partial_{23} & 0 & 0\\ 0 & -\partial_{13} & 0\\ 0 & 0 & -\partial_{12}\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix},$$

No. 20. (4,5,6; 1,2,3) (Morera solution)

$$\begin{bmatrix}\sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\\ \sigma_1\\ \sigma_2\\ \sigma_3\end{bmatrix} = \begin{bmatrix} -\partial_{11} & \partial_{12} & \partial_{13}\\ \partial_{12} & -\partial_{22} & \partial_{23}\\ \partial_{13} & \partial_{23} & -\partial_{33}\\ -2\partial_{23} & 0 & 0\\ 0 & -2\partial_{13} & 0\\ 0 & 0 & -2\partial_{12}\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix},$$

No. 2. (1,2,4; 3,5,6)

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_{23}\\ \sigma_3\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} 0 & \partial_{223} & \partial_{22}\\ -\partial_{33} & -\partial_{113} & \partial_{11}\\ \partial_{23} & \partial_{112} & 0\\ -\partial_{22} & 0 & 0\\ 0 & -\partial_{122} & 0\\ 0 & 0 & -\partial_{12}\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix},$$

No. 19. (3,5,6; 1,2,4)

$$\begin{bmatrix}\sigma_3\\ \sigma_{13}\\ \sigma_{12}\\ \sigma_1\\ \sigma_2\\ \sigma_{23}\end{bmatrix} = \begin{bmatrix} -\partial_{11} & \partial_{122} & 2\partial_{12}\\ \partial_{13} & -\partial_{223} & -\partial_{23}\\ 0 & \partial_{233} & \partial_{33}\\ -\partial_{33} & 0 & 0\\ 0 & -\partial_{133} & 0\\ 0 & 0 & -\partial_{13}\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix},$$

No. 3. (1,2,5; 3,4,6)

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_{13}\\ \sigma_3\\ \sigma_{23}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} -\partial_{33} & -\partial_{223} & \partial_{22}\\ 0 & \partial_{113} & \partial_{11}\\ \partial_{13} & \partial_{122} & 0\\ -\partial_{11} & 0 & 0\\ 0 & -\partial_{112} & 0\\ 0 & 0 & -\partial_{12}\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix},$$

No. 18. (3,4,6; 1,2,5)

$$\begin{bmatrix}\sigma_3\\ \sigma_{23}\\ \sigma_{12}\\ \sigma_1\\ \sigma_2\\ \sigma_{13}\end{bmatrix} = \begin{bmatrix} \partial_{112} & -\partial_{22} & 2\partial_{12}\\ -\partial_{113} & \partial_{23} & -\partial_{13}\\ \partial_{133} & 0 & \partial_{33}\\ -\partial_{233} & 0 & 0\\ 0 & -\partial_{33} & 0\\ 0 & 0 & -\partial_{23}\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix},$$

No. 4. (<u>1,2,6</u>; 3,4,5)

$$|A_1'| = 0,$$

No. 17. (3,4,5; 1,2,6)

$$\begin{bmatrix}\sigma_3\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_1\\ \sigma_2\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} -\partial_{11} & -\partial_{22} & -2\partial_{12}\\ 0 & \partial_{23} & \partial_{13}\\ \partial_{13} & 0 & \partial_{23}\\ -\partial_{33} & 0 & 0\\ 0 & -\partial_{33} & 0\\ 0 & 0 & -\partial_{33}\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix},$$

No. 5. (1,3,4; 2,5,6)

$$\begin{bmatrix}\sigma_1\\ \sigma_3\\ \sigma_{23}\\ \sigma_2\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} 0 & \partial_{33} & \partial_{233}\\ -\partial_{22} & \partial_{11} & -\partial_{112}\\ \partial_{23} & 0 & \partial_{113}\\ -\partial_{33} & 0 & 0\\ 0 & -\partial_{13} & 0\\ 0 & 0 & -\partial_{133}\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix},$$

No. 16. (2,5,6; 1,3,4)

$$\begin{bmatrix}\sigma_2\\ \sigma_{13}\\ \sigma_{12}\\ \sigma_1\\ \sigma_3\\ \sigma_{23}\end{bmatrix} = \begin{bmatrix} -\partial_{11} & \partial_{133} & 2\partial_{13}\\ 0 & \partial_{223} & \partial_{22}\\ \partial_{12} & -\partial_{233} & -\partial_{23}\\ -\partial_{22} & 0 & 0\\ 0 & -\partial_{122} & 0\\ 0 & 0 & -\partial_{12}\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix}, \qquad (22)$$

No. 6. (<u>1, 3, 5</u>; 2, 4, 6)

$$|A_1'| = 0,$$

No. 7. (1, 3, 6; 2, 4, 5)

$$\begin{bmatrix} \sigma_1 \\ \sigma_3 \\ \sigma_{12} \\ \sigma_2 \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} -\partial_{22} & -\partial_{233} & \partial_{33} \\ 0 & \partial_{112} & \partial_{11} \\ \partial_{12} & \partial_{133} & 0 \\ -\partial_{11} & 0 & 0 \\ 0 & -\partial_{113} & 0 \\ 0 & 0 & -\partial_{13} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix},$$

No. 8. (1, 4, 5; 2, 3, 6)

$$\begin{bmatrix} \sigma_1 \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_2 \\ \sigma_3 \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \partial_{223} & -\partial_{33} & 2\partial_{23} \\ \partial_{112} & 0 & \partial_{11} \\ -\partial_{122} & \partial_{13} & -\partial_{12} \\ -\partial_{113} & 0 & 0 \\ 0 & -\partial_{11} & 0 \\ 0 & 0 & -\partial_{13} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix},$$

No. 9. (1, 4, 6; 2, 3, 5)

$$\begin{bmatrix} \sigma_1 \\ \sigma_{23} \\ \sigma_{12} \\ \sigma_2 \\ \sigma_3 \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} -\partial_{22} & \partial_{233} & 2\partial_{23} \\ 0 & \partial_{113} & \partial_{11} \\ \partial_{12} & -\partial_{133} & -\partial_{13} \\ -\partial_{11} & 0 & 0 \\ 0 & -\partial_{112} & 0 \\ 0 & 0 & -\partial_{12} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix},$$

No. 10. (1, 5, 6; 2, 3, 4)

$$\begin{bmatrix} \sigma_1 \\ \sigma_{13} \\ \sigma_{12} \\ \sigma_2 \\ \sigma_3 \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} -\partial_{22} & -\partial_{33} & -2\partial_{23} \\ 0 & \partial_{13} & \partial_{12} \\ \partial_{12} & 0 & \partial_{13} \\ -\partial_{11} & 0 & 0 \\ 0 & -\partial_{11} & 0 \\ 0 & 0 & -\partial_{11} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix},$$

No. 15. (2, 4, 6; 1, 3, 5)

$$\begin{bmatrix} \sigma_2 \\ \sigma_{23} \\ \sigma_{12} \\ \sigma_1 \\ \sigma_3 \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} -\partial_{11} & -\partial_{33} & -2\partial_{13} \\ 0 & \partial_{23} & \partial_{12} \\ \partial_{12} & 0 & \partial_{23} \\ -\partial_{22} & 0 & 0 \\ 0 & -\partial_{22} & 0 \\ 0 & 0 & -\partial_{22} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix},$$

No. 14. (2, 4, 5; 1, 3, 6)

$$\begin{bmatrix} \sigma_2 \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_1 \\ \sigma_3 \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \partial_{133} & -\partial_{33} & 2\partial_{13} \\ -\partial_{112} & \partial_{23} & -\partial_{12} \\ \partial_{122} & 0 & \partial_{22} \\ -\partial_{223} & 0 & 0 \\ 0 & -\partial_{22} & 0 \\ 0 & 0 & -\partial_{23} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix},$$

No. 13. (2, 3, 6; 1, 4, 5)

$$\begin{bmatrix} \sigma_2 \\ \sigma_3 \\ \sigma_{12} \\ \sigma_1 \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} -\partial_{11} & \partial_{33} & -\partial_{133} \\ 0 & \partial_{22} & \partial_{122} \\ \partial_{12} & 0 & \partial_{233} \\ -\partial_{22} & 0 & 0 \\ 0 & -\partial_{23} & 0 \\ 0 & 0 & -\partial_{223} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix},$$

No. 12. (2, 3, 5; 1, 4, 6)

$$\begin{bmatrix} \sigma_2 \\ \sigma_3 \\ \sigma_{13} \\ \sigma_1 \\ \sigma_{23} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} 0 & \partial_{33} & \partial_{133} \\ -\partial_{11} & \partial_{22} & -\partial_{122} \\ \partial_{13} & 0 & \partial_{223} \\ -\partial_{33} & 0 & 0 \\ 0 & -\partial_{23} & 0 \\ 0 & 0 & -\partial_{233} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix},$$

No. 11. (<u>2, 3, 4</u>; 1, 5, 6)

$$|A_1'| = 0.$$

As we have proved above, each of the 17 forms of stress representations via three stress functions $\varphi_i$ is the general solution of the equilibrium equations (1) and, evidently, all these forms are equivalent to each other, i.e., any of them yields all solutions of Eqs. (1).

The Maxwell (No. 1) and Morera (No. 20) solutions have been known since the last century, whereas the remaining solutions (22) are new. The Beltrami–Krutkov–Blokh representations [3, 4] of stresses via the so-called tensor of stress functions in the form (2) are precisely the sum of the Maxwell (No. 1) and Morera (No. 20) solutions. But since any of the solutions (22) is general, as was proved, and Eqs. (1) are linear, the summation of these solutions does not yield any additional generality or completeness. In view of this, the attempts in many studies (see, e.g., [2–5, 10, 12, 13] to prove the generality or completeness of solution (2) are not, according to the above considerations, quite valid.

Hence, it follows from (22) that the stresses are expressed via stress functions in the form $\sigma = B\varphi$, where $B$ are the matrices that are presented in (22). However, on the other hand, $\sigma = AC'u$ [see (17)], i.e.,

$$B\varphi = AC'u, \tag{23}$$

the following relations being valid:

$$CB = 0, \qquad B'C' = 0. \tag{24}$$

Using Eqs. (17), (23), and (24), one can pose the elastic problem in terms of displacements and stresses to consider (23) as six equations for six functions $\varphi_1$, $\varphi_2$, $\varphi_3$, $u_1$, $u_2$, and $u_3$. With allowance for (17) and (24), from (23) we obtain the following equations in terms of displacements (see [26]):

$$CB\varphi = CAC'u = 0, \tag{25}$$

the compatibility conditions [see (11)]

$$B'C'u = B'\varepsilon = 0, \tag{26}$$

the stress equations

$$\varepsilon = A^{-1}\sigma \to B'A^{-1}\sigma = 0, \tag{27}$$

and the equations for stress functions

$$\sigma = B\varphi \to B'A^{-1}B\varphi = 0. \tag{28}$$

We write Eqs. (17) and (27) in terms of stresses (cf. [11, 26, 29])

$$C\sigma = 0, \qquad B'A^{-1}\sigma = 0 \tag{29}$$

or Eqs. (17) and (27) in terms of strains

$$CA\varepsilon = 0, \qquad B'\varepsilon = 0. \tag{30}$$

It is clear in (29) and (30) that there are six equations for six unknowns $\sigma_i$ or $\varepsilon_i$, rather than nine equations, as in the traditional Beltrami–Mitchell system for isotropic material.

As is seen from (22), the matrices $B$ and $B'$ have 17 possible forms. In (29) and (30), one can use any of these forms, i.e., 17 forms of Eqs. (28) in terms of stresses and strains can exist as well. It is possible to substitute $B'$ and $B$ of different forms into Eqs. (28), i.e., we obtain $17 \cdot 17 = 289$ versions of equations for the stress functions $\varphi_i$. However, since Eqs. (17) and (25)–(30) are reduced to (23), it is, apparently, expedient to proceed from system (23) in concrete problems.

Borodachev [27, 28] proved that the first three Saint Venant compatibility conditions and the next three [see (12)] are derived from one another by replacement of differentiation symbols by the parameters of the Fourier transform. This corresponds to a particular case of the results of the present study.

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